ECONOMETRICS OF FIRST PRICE AUCTIONS: A SURVEY OF THE THEORETICAL AND APPLIED LITERATURE

FLORENCIA GABRIELLI

RESUMEN

Analizar mercados de subastas comprende un área activa de investigación, tanto en el plano teórico como empírico. Este trabajo contiene una revisión de la literatura relevante para realizar econometría de subastas desde la perspectiva del enfoque estructural. El enfoque estructural es un marco para analizar datos de subastas en el cual la teoría y la econometría están estrechamente vinculadas. Hay tres aspectos fundamentales para el enfoque estructural: (i) Existen restricciones testeables impuestas desde el modelo teórico sobre los datos? (ii) Es posible identificar a los elementos estructurales que caracterizan una subasta sin tener que recurrir a información paramétrica a priori? (iii) Que puede decirse sobre procedimientos implementables de estimación que no se basen en supuestos paramétricos? Esta recopilación muestra como la literatura sobre el enfoque estructural ha respondido a cada una de estas preguntas.
Clasificación JEL: C14, C70, D44, L10, L41
Palabras Clave: Subastas, Enfoque Estructural, Colusión, Estimación No Paramétrica, Polinomios Locales.

ABSTRACT

The analysis of auctions is an active area of research for both theoretical and empirical economists. This paper provides an overview of the literature relevant to the econometrics of auctions from a structural approach perspective. The structural approach is a framework to analyze auction data in which theory and econometrics are closely related. There are three fundamental issues at the heart of the structural analysis: (i) Are there any testable restrictions imposed by the theoretic model on the data? (ii) Is it possible to identify the structural elements that characterize an auction without a priori parametric information? (iii) What can be said about feasible estimation procedures which do not rely on parametric assumptions?. This survey shows how the structural approach literature has answered each question.
JEL Classification: C14, C70, D44, L10, L41
Keywords: Auctions, Structural Approach, Collusion, Nonparametric Estimation, Local Polynomial Fitting.

ECONOMETRICS OF FIRST PRICE AUCTIONS: A SURVEY OF THE THEORETICAL AND APPLIED LITERATURE[*]

FLORENCIA GABRIELLI[*]

I. Introduction

The analysis of auctions is an active area of research for both theoretical and empirical economists. This survey attempts to summarize the state of the literature in Auction Theory as well as the Econometric Methods mainly used to estimate the theoretical models. In particular, since most of the data sets available for empirical research come from first-price auctions mechanisms, I mainly concentrate in this auction type.

Auctions and procurements are widely used mechanisms to allocate public contracts, financial assets, agricultural products, natural resources, artwork and electricity, to name a few. Also, auctions throughout the internet has become quite relevant in the last few years. Given the extensive use of auctions to allocate goods and services which are frequently public, many data sets are available for empirical research.

Auctions have well defined rules that can be used to build a transparent economic model. The Bayesian Nash Equilibrium (BNE) concept allows to model auctions as a game in which a buyer/seller faces a limited number of participants that behave strategically. Auctions are typically modeled as games of incomplete information in which the asymmetry of information between the buyer/seller and the participants and among the participants themselves play a key role.

Until recently, the empirical analysis of auction data was limited to test some predictions from the underlying game theoretical models (see Porter (1995) for an exhaustive survey on the Reduced Form Approach). This kind of analysis is known in the literature as the Reduced Form Approach. This approach has the drawback that does not allow to do policy evaluation that would require knowledge of the informational structure of the game, like the

[*] I am grateful to Joris Pinkse for very insightful comments and guidance.
[*] CONICET – UNCUYO.
E-Mail: orgabrielli@gmail.com. Address: Pizzurno 165 Godoy Cruz-Mendoza, Argentina.

optimal choice of the reserve price or the auction mechanism that would generate a greater revenue for the buyer/seller.

On the other hand, the Structural Approach allows the researcher to do policy evaluation by assuming that observed bids are the equilibrium bids from some auction model. Specifically, $b_i = s_i(v_i)$, where b_i denotes the (observed) bid made by player i, $s(.)$ is the equilibrium strategy and v_i is the (unobserved) private value that player i has for the auctioned item. It is assumed that private information of the players comes from some distribution, which is common knowledge for all participants. This distribution and the individual preferences are the key elements that explain the behavior, which are the structure of the model. In other words these are the structural elements from the econometric model induced by observed bids.

The Structural Approach then exploits the equilibrium relation $b_i = s_i(v_i)$ to recover private information from participants, which then can be used to make policy recommendations.

As recognized by Guerre, Perrigne and Vuong (2000) an important difficulty associated to the Structural Approach is its numerical complexity and in the implicit form of the equilibrium strategy. There are three aspects that have to be taken into account. In the first place there is an Identification issue, that is if the structure of the auction can be recovered univocally from observable variables (bids) minimizing parametric restrictions. In other words, identification implies establishing if the corresponding models can be discriminated from observable variables. A second important issue is model validity that is if the theoretical model imposes testable restrictions on the data. Finally, a third difficulty is associated to the development of tractable estimation methods.

In their seminal paper about structural analysis of auctions, Laffont and Vuong (1996) highlight that there are practical and theoretical reasons that have lead to an important development of Econometrics of Auctions. As mentioned above, there are important data sets that can be used, given that these mechanisms are extensively applied in the real work. On the other hand significant theoretical contributions to Industrial Organization have been done and now the challenge is to take such contributions to applied work.

The paper is organized as follows. Section 2 deals with the theoretical models developed in the literature. Section 3 describes the econometric methods available to estimate the models previously detailed. In particular in this section I discuss in detail the "so called"' Structural Approach for analyzing auction data and estimating auction models. Section 4 describes kernel methods and local polynomial fitting, two methods extensively used in the estimation of auction models. Section 5 analyzes how to incorporate cooperative behavior (e.g. collusion) in the models and how the econometric strategy can be adapted. Finally Section 6 collects the main conclusions.

II. Auction theory

In this section I present a unifying view of game-theoretical models of first-price auctions from a structural econometric perspective.[1] This framework has been analyzed by Laffont and Vuong (1996) following the theoretical contribution of Wilson (1977). At a theoretical level the most general paradigm identified in the literature to model auctions is the Affiliated Value (AV) model which is defined by the pair $[U(.), F(.)]$, where $U(.)$ represents the utility function and $F(.)$ denotes the distribution of the information bidders have. This information could be private to each bidder or common to all bidders. More precisely, $U_i(v_i, c)$ is the utility of a potential bidder i, $i = 1, \ldots, n$ for the object where $\$v_i$ denotes the ith player's private signal or information and c represents a common component or value affecting all utilities. The utility, $U(.)$, is increasing in both arguments. The vector $(v_1, \ldots, v_n, c)$ is a realization of a random vector whose $(n+1)$ dimensional cumulative distribution function is $F(.)$. The latter is assumed to be affiliated with a support $[\underline{v}, \overline{v}]^n$x $[\underline{c}, \overline{c}], \underline{v} \geq 0$, and a density $f(.)$.[2] Each bidder i knows the value of his signal, v_i, but neither the other signals v_j nor the common value v. On the other hand, the number n and the functions $U(.)$ and $F(.)$ are common knowledge.

Depending on the nature of the utility function and that of the information held by bidders, a further sub classification produces the Affiliated Private Value (APV) paradigm in which $U(v, c) = v$, and the General Common Value

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[1] For other mechanisms to model auctions such as second price, Dutch and English auctions the reader is refereed to Krishna (2002).

[2] Roughly speaking, affiliation is a strong form of positive correlation. In an auction context, private signals $V_1, \ldots, V_n$ are affiliated if when a subset of the V_i's are all large, then this makes it more likely that the remaining V_j's are also large. For a formal definition see Krishna (2002).

paradigm, where $U(v,c) = c$. Each model in turn is said to be symmetric if the function $F(.)$ is symmetric in its first N arguments, otherwise the models are called asymmetric. A special case of the APV model is the Independent Private Value (IPV) model. In the case of symmetric bidders, this model takes the form $U_i = v_i$ with V_i independently and identically distributed as $F(.)$. For the asymmetric IPV model, the utility function is still given by $U_i = v_i$, but (in its simplest form) one bidder draws his valuation from a distribution that differs from that of other bidders.

II.1. Symmetric AV Model

Among others, Wilson (1977) and Milgrom and Weber (1982) have characterized the Symmetric AV model. From this literature it is known that the Nash Equilibrium of this game is strictly increasing and is obtained from the following optimization problem.

$$\max E[(U(v_i, c)) - b_i 1\!\!1(B_i \leq b_1)|v_i]$$

With $B_i = s(y_i)$, $y_i = max_{j \neq i} v_j$, $s(.)$ is the equilibrium strategy. It can be shown that the solution is given by,

$$b_i = s(v_i) = V(v_i, v_i) - \int_{\underline{v}}^{v_i} L(\alpha|v_i) dV(\alpha, \alpha) \tag{1}$$

where $L(\alpha|v_i) = \exp[-\int_{\alpha}^{v_i} f_{y_1|v_1}(s|s)/F_{y_1|v_1}(s|s)ds]$ and $V(v_i|y_i) = E[U(v_i, c)|v_i, y_i]$, the boundary condition is given by $s(\underline{v}) = V(\underline{v}, \underline{v})$.

As can be seen in this case there is a closed form solution for the equilibrium.

There are two special cases that have been extensively studied. In the first place the symmetric model with independent private values, the IPV model ($U_i = v_i$ and v_i iid as $F(.)$) (see Vickrey (1961), Riley and Samuelson(1981)). In the second place the symmetric common value model, the CV model

($U_i = c$ with v_i iid given c as $F_{v|c}(.|.)$). In any of these two cases the equilibrium (1) simplifies (see Rothkopf (1969) and Wilson (1969)).

II.1.1. Symmetric IPV Model

In this paradigm bidders obtain their private valuations independently from a common distribution, $F(.,\dots,.)$. In particular $F(.,\dots,.) = F(.)^n$. The BNE is symmetric, given that the game is symmetric. Therefore each player uses the same strategy $(.)$: $[\underline{v}, \overline{v}] \to [\underline{b}, \overline{b}]$.

This model has been extensively studied (see Riley and Samuelson (1981)). Several important theoretical results have been obtained from this model (*Revenue Equivalence Theorem* Vickrey (1961) and *Optimal Reserve Price*).[3]

A single and indivisible object is offered for sale to n potential buyers who bid in an auction where the highest bidder gets the object and pays the amount of his bid. Each bidder i assigns a value V_i to the object, the maximum amount the bidder is willing to pay for the object. It is assumed that for each i, $i = 1,\dots,n$ is distributed according to the increasing distribution function $F(.)$ with support on $[\underline{v}, \overline{v}]$. $F(.)$ is common knowledge and admits a continuous density $f(.)$ on $[\underline{v}, \overline{v}]$. V_i is the private information of bidder i. In other words, bidder i observes a realization v_i of V_i and only knows that other bidders' valuations are independent draws from $F(.)$. The fact that for all i, $F_i(.) = F(.)$ is referred to as a situation involving symmetric bidders. Let s_i: $[\underline{v}, \overline{v}] \to \mathbb{R}_+$ denote bidder i's strategy, which determines his bid for any private value. Given that bidders are symmetric, it is natural to concentrate on symmetric equilibrium, i.e. $s_i = s$.

For a given bid, b_i, the payoffs for the ith bidder are given by

$$\Pi_i = \begin{cases} (v_i - b_i) & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j. \end{cases} \quad (2)$$

[3] *The Revenue Equivalence Theorem* establishes that in the IPV paradigm ascending, descending, first-price and second-price auctions have associated the same expected revenue for the seller.

where in case of a tie among two or more bidders it is assumed that the object goes to each bidder with equal probability. From the expected profit function (2) above it can be seen that no bidder would bid an amount equal to his value since in this case the payoff would be zero. Thus, in equilibrium bidders shade their valuations. Notice also that a bidder faces a trade-off at any bid holding constant the behavior of his rivals. Indeed, by increasing one's bid, the probability of winning also increases, but at the same time there is a reduction in the gains from winning. More formally, the optimization problem for each bidder is

$$\max_{b_i} (v_i - b_i) F(s^{-1}(b_i))^{n-1}$$

where $s^{-1}(.)$ denotes the inverse strategy function and $F(s^{-1}(b_i))^{n-1}$ is the probability of winning.

Riley and Samuelson (1981) among others have characterized the unique symmetric differentiable Bayesian Nash equilibrium. When $n \geq 2$ the solution is unique and from the first order conditions (FOC) one has,

$$b_i = s(v_i, F, n) \equiv v_i - \frac{1}{[F(v_i)]^{n-1}} \int_{\underline{v}}^{v_i} [F(u)]^{n-1} du. \tag{3}$$

subject to the boundary condition $s(\underline{v}) = \underline{v}$. Then the solution can be written as,

$$1 = [v_i - s(v_i)](n-1) \frac{f(v_i)}{F(v_i)} \frac{1}{s'(v_i)} \tag{4}$$

The expression in (3) shows that the equilibrium bid is less than the private value. Moreover, the degree of "shading" (the amount by which the bid is less than the value) depends on the number of competing bidders. In particular, the larger the number of competing bidders, the smaller the difference between bids and private values (i.e. the degree of shading is less in auctions with more participating bidders).

II.1.2. Symmetric APV Model

A weakness of the IPV model is the restrictive assumption of independent values. A more general framework is therefore given by the APV model. In this case, bidders draw their private valuations from a joint and affiliated distribution $F(.,\dots,.)$. Let $y_i = max_{j\neq i} v_i$, the equilibrium strategy satisfies,

$$s'(v_i) = [v_i - s(v_i)]\frac{f_{y1|v1}(v_i|v_i)}{F_{y1|v1}(v_i|v_i)},$$

for all $v_i \in [\underline{v}, \overline{v}]$, subject to the boundary condition $s(\underline{v}) = \underline{v}$. $F_{y_1|v_1}(.|.)$ denotes the conditional distribution of y_1 given v_1 and $f_{y_1|v_1}(.|.)$ is the conditional density. The index 1 refers to any player given that all are ex-ante identical. From Milgrom and Weber (1982) it is known that the solution is,

$$b_i = s(v_i, F) = v_i - \int_{\underline{v}}^{v_i} L(\alpha|v_i)d\alpha, \tag{5}$$

with $L(\alpha|v_i) = \exp[-\int_{\alpha}^{v_i} \frac{f_{y1|v1}(u|u)}{F_{y1|v1}(u|u)} du]$.

This model with more reliable assumptions than the IPV model has also a wide potential for empirical applications.

II.2. Asymmetric AV Model

Asymmetries introduce numerous complications in first--price auctions. In particular, and despite the fact that equilibrium of this game exists under some regular conditions, there is no closed form expression for the bidding strategies. This further complicates the econometric analysis of this kind of auctions. I explain in section 3 how the use of indirect methods of estimation is especially useful in this context. Another feature of the asymmetric first--price auction model is that the resulting allocation is not necessarily efficient; that is, the object may not end up in the hands of the bidder who values it the most.[4]

The general theory is less developed for this kind of models. Asymmetries can arise because some players could be better informed than others, also because some players maybe better organized through cartels, other source of asymmetries are different characteristics of participants like size and location. Three special cases will be considered.

1. Asymmetric IPV Model
2. Asymmetric CV Model
3. Asymmetric APV Model

II.2.1. Asymmetric IPV Model

In this case $U_i = v_i$,and I am going to consider that $V_1 \sim F_1$, $v_j \sim F_2(.)$, $F_1 \neq F_2$. It is clear that the problem faced by player 1 is different than the one faced by the rest of the bidders.

A limitation of symmetric models is the assumption that all bidders are ex-ante identical, which may not be the case in many situations that are observed in empirical applications. To simplify the exposition I assume that there are two types of bidder. The model can easily be extended to a situation with more than two types.[5] For instance, group 1 could be the one containing better informed bidders or bigger size bidders, etc. Group 2 gathers all other bidders. Let n_j be the number of players of type j, $j = 1{,}2$ with $n_1 + n_2 = n \geq 2$. The model assumes that inside each group or type players are ex-ante identical, that is their valuations are iid random variables from $F_j(.)$. Therefore $F(.,\dots,.) = F_1(.)^{n_1} F_0(.)^{n_2}$. Each one of these distributions are common knowledge and have common support, $[\underline{v}, \overline{v}]$.[6] The corresponding densities are $f_1(.)$ and $f_2(.)$ which are assumed to be continuously differentiable and bounded away from

[4] As is well known in the auction theory literature, the Revenue Equivalence Principle holds under the assumption of symmetry (see see Krishna (2002) for further details).
[5] For an application with three types of bidder see for example Aryal and Gabrielli (2013).
[6] The assumption of a common support is for simplicity.

zero on their support.[7] The equilibrium strategies $s_1(.)$ and $s_2(.)$ have in general no closed form. Thus the use of numerical methods becomes necessary, a major difficulty associated to these kinds of models. Moreover, assume that these strategies are increasing and differentiable with inverses $\xi_1 = {s_1}^{-1}$ and $\xi_2 = {s_2}^{-1}$. The expected payoff of a type 1 bidder, say, when a type 2 bidder follows the strategy $s_2(.)$ is

$$\begin{aligned}\Pi_1(b, v_1) &= F_2(\xi_2(b))(v_1 - b) \\ &= G_2(b)(v_1 - b)\end{aligned} \tag{6}$$

where $G_2(.) \equiv F_2(\xi_2(.))$ denotes the distribution of type 2 bidders' bids.

The equilibrium strategies $s_1(.)$ and $s_2(.)$ do not have closed form solution in general. Nevertheless, these strategies satisfy the following system of differential equations.[8]

$$\begin{aligned}s_1'(v_{1i}) &= [v_{1i} - b_{1i}]\left[(n_1 - 1\frac{f_1(v_{1i})}{F_1(v_{1i})}) + n_0\frac{f_0(s_0^{-1}(b_{1i}))}{F_0(s_0^{-1}(b_{1i}))}\frac{s_1'(v_{1i})}{s_0'(s_2^{-1}(b_{1i}))}\right] \\ s_2'(v_{2i}) &= [v_{2i} - b_{2i}]\left[(n_2 - 1\frac{f_2(v_{2i})}{F_0(v_{2i})}) + n_1\frac{f_1(s_1^{-1}(b_{2i}))}{F_1(s_1^{-1}(b_{2i}))}\frac{s_2'(v_{2i})}{s_1'(s_1^{-1}(b_{2i}))}\right]\end{aligned}$$

subject to the boundary conditions $s_2(\underline{v}) = s_1(\underline{v})$ and $s_1(\overline{v}) = s_2(\overline{v})$.

II.2.2. Asymmetric CV Model

Assume that one bidder is fully informed and the remaining are uninformed. $U_i = c$ for all players with $v_1 = c$ and $v_j = \emptyset$ for $j \neq 1$. In addition, $c \sim F_c(.)$. The strategy of the informed bidder is,

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[7] This set of regularity conditions guarantees the existence and uniqueness of the equilibrium. See, e.g Lebrun (1996, 1999), and Maskin and Riley (2000a,b, 2003).

[8] To see an example which uses Uniform distributions see Krishna (2002).

$$b_1 = s_1(v_1) = v_1 - \int_{\underline{v}}^{v_1} \frac{F_c(u)}{F_c(v_1)} du \tag{6}$$

Non informed bidders should adopt mixed strategies $M(.)$ in the interval $[v, E(c)]$, so that the maximum bid mimics the distribution of the informed bidder, that is,

$$M^{n-1}(\cdot) = F_c[s^{-1}(\cdot)]$$

II.2.3. Asymmetric APV Model

In this case and under the assumption that there are two types of bidders this model considers that the vector $(v_{11}, \ldots, v_{1n_1}, v_{01}, v_{0n_0})$ is distributed according to the joint distribution $F(.)$. This function is assumed to be exchangeable in its first n_1 components and also in its last n_0 components. The intuition behind this probabilistic structure is that inside a group or type players are ex-ante identical. Moreover, since $F(.)$ is affiliated there is general dependence among private values. Campo, Perrigne, and Vuong (1998) have characterized the system of differential equations that define the equilibrium of this game. Given that this model belongs to the asymmetric paradigm there is no closed form solution.

II.3. Identification and Testeable Restricctions: the symmetric case

The main hypothesis behind the Structural Approach is that observed bids are the equilibrium bids of the auction model under consideration. Given n and the structure $[U(.), F(.)]$, the equilibrium strategy of the game induces the following econometric model,

$$b_i = s_i(v_i, n, U, F).$$

It is important to notice that v_i is a random variable which is non-observed but is distributed according to $F(.)$. Therefore b_i are also random variables distributed according to the function $G(.)$ that is determined by the structural elements of the model. In this sense, the function $G(.)$ depends on $F(.)$ through two different channels:

1. The $v's$, given that $v_i \sim F(.)$.

2. Through the equilibrium strategy, $s(.)$ (see e.g. (4))

These characteristics that are proper to auction models further complicate the identification and characterization of theoretical restrictions on data. Thus, in order to be able to establish "Identification" one takes n and observed bids as given, that is this is the only available information for the research to identify the model.[9]

First, it is necessary to precise what is being understood by models that can not be identified or distinguish one from another. The following definition intents to clarify this issue.

Definition: two models M and $\widetilde{M}$ are observationally equivalent given the (observed) bids $(b_1, \dots, b_n)$ if both rationalize the same equilibrium bid distribution, that is, $G(.) = \tilde{G}(.)$.

The seminal work of Laffont and Vuong (1996) establishes a series of results about the identification (or lack of it) for the models outlined the previous section. Here I replicate these results and give some intuition about them.

Proposition 1: Any symmetric AV model is observationally equivalent to some symmetric APV model. Thus a symmetric AV model is unidentified, in general.

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[9] This framework is quite realistic since typically the only available information are observed bids and the number of bidders, besides observed heterogeneity that can easily be incorporated in the analysis.

The proof of this result (not given) is based on the following key observation. $U_i = U(v_1, c)$ can always be replaced by $\widetilde{U}_i = \widetilde{v}_i$ with $\widetilde{v}_i = V(v_i, v_i)$. In other words, dependence among utilities given by the common component c, can be replaced by (appropriate) dependence among private signals. Therefore, without loss of generality it is enough to concentrate in the identification of the APV model ($U_i = v_i$), given that this constitutes the most general model that can be identified.

The next result establishes that the IPV model is indeed identified from observed bids.

Proposition 2: A symmetric APV model is identified.

The intuition behind this result can be expressed as follows: given that (observed) bids are related with private values through the equilibrium strategy which is strictly increasing, then $F(.)$ can be identified from $G(.)$.

Proposition 3: A symmetric IPV model is identified.

Guerre, Perrigne, and Vuong (2000) contains a formal proof of this statement. The intuition behind this result is similar to that behind Proposition 2 above.

As Guerre, Perrigne, and Vuong (2000) emphasize, the equilibrium relation that links the observed bid b_i to the underlying private value v_i is strictly monotonic which implies that the identification problem is non-trivial. The difficulty associated to the identification problem relies on the fact that the distribution $G(.)$ of b_i depends on the underlying distribution $F(.)$ in two ways: directly through v_i, which is distributed as $F(.)$, and indirectly through the equilibrium strategy $s(.)$, which depends on $F(.)$ (see (4)).

Theorem 1 in Guerre, Perrigne, and Vuong (2000) solves the identification problem by stating that the distribution $F(.)$ is unique whenever it exists. In addition, it gives a necessary and sufficient condition on the distribution $G(.)$

for the existence of a distribution $F(.)$ of bidders' private values that can rationalize $G(.)$.

This result relies upon the fact that the first derivative $s'(.)$ and the distribution $F(.)$ with its density $F(.)$ can be eliminated at the same time from the differential equation (4) by introducing the distribution $G(.)$ of b_i and its density $g(.)$. Specifically, for every $b \in [\underline{b}, \overline{b}] = [\underline{v}, \mathrm{s}(\overline{v})]$ one has that $G(b) = Pr(B \leq b) = Pr(V \leq s^{-1}(b)) = F(s^{-1}(b)) = F(v)$, where the last equality uses $b = s(v)$. It follows that the distribution $G(.)$ is absolutely continuous with support $[\underline{v}, \mathrm{s}(\overline{v})]$ and density $g(b) = f(v)/s'(v)$, where $v = s^{-1}(b)$. Taking the ratio gives $g(b)/G(b) = (1/s'(v))\, f(v)/F(v)$. Thus the differential equation (4) becomes

$$v_i = \xi(b_i, G, n) \equiv b_i + \frac{1}{n-1}\frac{G(b_i)}{g(b_i)}. \tag{8}$$

Equation (8) shows the individual private value v_i as a function of the individual's equilibrium bid b_i, its distribution $G(.)$, its density $g(.)$ and the number of bidders n.

In other words, (8) states that if b_i is the equilibrium bid, as it is assumed in the structural approach, then the bidder's private value v_i corresponding to b_i must satisfy (8).

It is important to highlight that for the identification argument outlined above it is not necessary to solve the FOC of the underlying theoretic model. In particular, the identification of the structural model does not require a priori parametric specifications. Moreover, because it is nonparametric in nature, this identification result applies to parametric identification as well.

Another important aspect to note is that the function $\xi(., G, n)$ is completely determined from the knowledge of $G(.)$ and n. Because $\xi(., G, n)$ is the quasi inverse of $s(., F, n)$, one has neither to solve the differential equation (8) nor to apply numerical integration in (3) so as to determine the buyers' equilibrium strategy $s(., F, n)$. This remark is important because it underlies the principle and the computational advantages of the indirect estimation method proposed by Guerre, Perrigne, and Vuong (2000).

Regarding CV models, the general result is that they are not identified, which arises from Proposition 1.

Proposition 4: A symmetric CV model is unidentified, in general.

To better understand the intuition behind this result one has to observe the fact that if new signals $\widetilde{v_i}$ are defined as strictly increasing transformation of the original signals v_i, then it is clear that the identification of this model is not possible.[10]

The results stated above establish the identification of some of the models in the symmetric case, and the lack of identification in other cases. The next step is to see if there are testable restrictions from theory on the data. To this end, I maintain the assumption that $F(.)$ is symmetric and affiliated. Given that $b_i = s_0(v_i)$, $\forall i$ it follows that $G(.)$ is also symmetric and affiliated in $[\underline{b}, \overline{b}]^n$.

The distribution $G(.)$ of the bids $(b_1, \dots, b_n)$ is rationalized by a structure $[U(.), F(.)]$ if $G(.)$ is the equilibrium distribution of the corresponding game.

Proposition 1 implies that the restrictions imposed on an AV model are the same that can be imposed on an APV model (beyond symmetry and affiliation). Define the following function that is needed for the next result. Let $G_{\{B_1|b_1\}}(.|.)$ be the conditional distribution of $B_1 = max_{j \neq 1}\, b_j$ given b_1.

Proposition 5: A distribution $G(.)$ can be rationalized by a symmetric APV model if and only if the function

$$\xi(b) = b + \frac{G_{B_1|b_1}(b|b)}{g_{B_1|b_1}(b|b)}$$

is strictly increasing in $b \in [\underline{b}, \overline{b}]$.

[10] Recall that the CV model has been characterized within the unified framework used in this paper as $U_i = c$ with v_i *iid* given c, see section 2.1.

It is important to highlight that $s_0(.)$(the strategy actually used by bidders) might not be the same as $s(.)$ (the equilibrium strategy) that is obtained from the FOC of the game. Thus, in order to test if bidders behave as predictive by game theory one has to corroborate if $\xi(b)$ is strictly increasing, otherwise $s_0(.)$ can not be a Bayesian Nash equilibrium strategy. To prove this proposition one needs to use the FOC of the game and a similar argument to that in Guerre, Perrigne, and Vuong (2000).

The next result concerns IPV models, which are a special case of APV models. Thus, the testable restrictions imposed by theory are readily characterized. Guerre, Perrigne, and Vuong (2000) contains a formal proof of this argument.

Proposition 6: A distribution $G(.)$ can be rationalized by a symmetric IPV model if and only if

1. $G(b_1,..,b_n) = \prod_{i=1}^{n} G_0(b_i)$ **for some distribution** $G_0(.)$ **with support** $[\underline{b}, \overline{b}]$**; and**

2. the function $\xi(b) = b + \frac{1}{n-1}\frac{G_0(b)}{g_0(b)}$ **is strictly increasing in** $b \in [\underline{b}, \overline{b}]$**.**

The first part says that valuations are independent and identically distributed (iid). The second part characterizes the equilibrium strategy and establishes the theoretical restriction that one should test.

As mentioned above, the state of the art is less developed for (symmetric) CV models, and consequently the restrictions coming from game theory are not fully available. More recently, there have been some results Hendricks, Pinkse, and Porter (2003), Haile, Hong, and Shum (2003)). However, it is clear that the equilibrium bids in a CV model should be dependent while in an IPV model they are independent. Therefore, in principle IPV and CV models can be distinguished.

II.4. Identification and Testeable Restricctions: the asymmetric case

In this section I present the main results about identification and testable restrictions coming from theory for the models characterized in section 2.2. The questions that these results address are the following.

- Are any of these models identified? (Propositions 7 and 8).
- Are there theoretical restrictions that each model imposes over the distributions of bids? (Propositions 7 and 8).
- Can these asymmetric models be distinguished from observed bids? (Proposition 9).
- It is possible to distinguish an asymmetric model from a symmetric one? (Proposition 10).

Proposition 7: The asymmetric IPV model of section 2.2.1 is identified. Moreover, a distribution $G(.)$ can be rationalized by such a model if and only if:

1- $G(b_1,.,b_n) = G_1(b_i)\prod_{i=2}^{n} G_2(b_i)$ **for some distributions** $G_1(.)$ **and** $G_2(.)$**; and**

2- the functions

$$\xi_1(b) = b + \frac{1}{n-1}\frac{G_2(b)}{g_2(b)}$$

$$\xi_2(b) = b + \frac{1}{\frac{g_1(b)}{G_1(b)} + (n-2)\frac{g_2(b)}{G_2(b)}}$$

are strictly increasing in $b \in [\underline{b}, \overline{b}]$**, the common support of** $G_1(.)$ **and** $G_2(.)$ **.**

The next result establishes that it is possible to identify the asymmetric CV model of section 2.2.2 and what are the theoretical restriction on it.

Proposition 8: The asymmetric CV model of section II.2.2. is identified. Moreover, a distribution $G(.)$**can be rationalized by such a model if and only if:**

1- $G(b_1,.,b_n) = G_1(b_i)\prod_{i=2}^{n} M(b_i)$ **for some distributions** $G_1(.)$ **and** $M(.)$**; and**

2- $G_1(.) = M^{n-1}(.)$ **and the function**

$$\xi(b) = b + \frac{1}{(n-1)}\frac{M(b)}{m(b)}$$

is strictly increasing in $b \in [\underline{b}, \overline{b}]$**, the common support of** $G_1(.)$ **and** $M(.)$**.**

Despite the fact that each model is identified it is valid to ask whether these two asymmetric models can be discriminated from observed bids. In order to answer the third question above I present the following result.

Proposition 9: Any asymmetric CV model is observationally equivalent to an asymmetric IPV model.

The proof of this result follows from using Propositions 7 and 8 and taking $G_2(.) = M(.)$and the same $G_1(.)$ in both cases.

Finally, in order to answer the last question above it is enough to consider the case $n = 2$. Then, the conditions of Propositions 8 and 6 are equivalent, leading to the following statement.

Proposition 10: Any asymmetric CV model need not be distinguishable forma a symmetric one.

III. Econometrics of Auctions

The availability of numerous data sets and the well-defined game forms associated with auctions mechanisms makes “econometrics of auctions” a particularly interesting field within economics. There are three main approaches to analyze auction data. The experimental approach, aimed to test the predictions of game-theoretic models, uses experimental data for controlling the underlying elements of the model. The reduced-form approach uses field data mostly relying on linear regressions of the logarithm of bids on a set of observed variables. These first two approaches were used almost exclusively until the late 1980s constituting the first step towards the formulation of empirical auction models. Most recently, a so--called structural approach has been developed. A detailed discussion and references to this literature is given by Perrigne and Vuong (1999, 2008) survey papers. In particular these authors concentrate on first-price auctions within the private value paradigm.[11]

III.1. Structural Approach to Analyze Auction Data

Assuming that observed bids are the Bayesian Nash equilibria of the game-theoretical model under consideration, the structural approach provides a rich framework in which the theoretical model and its empirical counterpart are closely related. The main objective of this approach is to recover the structural elements of the auction model. This line of research has attracted considerable attention over the last fifteen years.

A first classification of methods for structurally estimating auction models distinguishes between direct methods and indirect methods. In turn, these methods can be parametric, if some of the structural elements are assumed to be known except for a finite number of parameters. Also, non-parametric techniques can be used to estimate. In this case there are no assumptions about the functional form of the structural elements.

Relying on parametric econometric models, direct methods were first developed in the literature. The starting point is to specify the underlying distribution of private values in order to estimate the parameter vector characterizing such a distribution. Within this class of methods, there are two major estimation procedures. The first methodology introduced by Paarsch (1992) and Donald and Paarsch (1993) is a fully parametric setup that uses ML-based estimation procedures requiring the computation of the equilibrium

[11] See also Paarsch and Hong (2006) for an extensive survey on structural estimation of auction models within the IPV paradigm. Hendricks and Porter (2007) contain an exhaustive recompilation of examples within the Reduced Form Approach. On the other hand Athey and Haile (2007)have documented the use of nonparametric techniques to estimate structural auction models.

strategy. This in turn could be highly computationally demanding, as recognized by Donald and Paarsch (1993), and thus only very simple distributions are considered in practice. In particular, because the upper bound of the bid distribution depends on the parameter(s) of the underlying distribution, the ML estimator has a nonstandard limiting distribution. In view of this, Donald and Paarsch (1993) develop a so-called piecewise pseudo ML estimator requiring the computation of the equilibrium strategy that can be obtained using specific parametric distribution(s). Donald and Paarsch (1993) have established the asymptotic properties of this estimator, that is, they have shown its consistency and that it distribution converges to a normal distribution at the parametric $\sqrt{L}$ rate.[12] Marshall, Meurer, Richard, and Stromquist (1994) propose a set of numerical algorithms to solve for the equilibrium strategy of an asymmetric first-price auction allowing for arbitrary distributions of the private values.

In a second paper, Donald and Paarsch (1996) study the identification problem of the parameter θ, and also the properties of the ML estimator in the case in which the upper extreme of the support depends on the unknown parameter. The asymptotic distribution that they obtain for this estimator is non-standard which constitutes a drawback in practice. Both methods based on ML are subject to two major limitations. In the first place they only allow including observable heterogeneity (of auctions) through discrete variables. In the second place there is the great complexity to compute these estimators given that the associate routines are quite complex and require the calculation of the equilibrium strategy and its inverse to determine the upper bound of the bid distribution and density.

Laffont, Ossard, and Vuong (1995) introduced a second methodology, which is more computationally convenient. Relying on the Revenue Equivalence Theorem, the authors propose a simulation-based method that avoids computation of the equilibrium strategy and therefore allows for more general parametric specifications for the private value distribution. This method is called simulated nonlinear least squares (SNLLS) and is related to the methods proposed by McFadden (1989) and Pakes and Pollard (1989). The SNLLS estimator of Laffont, Ossard, and Vuong (1995) uses only the winning bid since the authors considered a descending oral auction of eggplants in their application. The authors establish the asymptotic properties of the estimator,

12 L denotes the number of auctions considered for estimation, i.e. the sample size.

namely its consistency and normality at the parametric $\sqrt{L}$ for a given number of simulations and the number of auctions goes to infinity. The main advantage of this method is that is not necessary to compute the equilibrium strategy or its inverse. This makes this method attractive from a computational point of view and also makes it usable with any parametric family of distributions $F(.)$.[13] Li and Vuong (1997) extended the SNLLS estimator to the case in which all bids are observed, such as in sealed-bid auctions.

The use of indirect methods to structurally estimate auction models was introduced in Guerre, Perrigne, and Vuong (2000). The authors develop a fully nonparametric procedure for the structural estimation of auction models. This alternative methodology relies on a simple but crucial observation, namely that each private value can be expressed as a function of the corresponding bid, the distribution of observed bids and its density using the first-order condition of the bidder's optimization problem. Thus, in contrast to direct methods, the starting point of indirect procedures is the distribution of (observed) bids, which is used to recover the distribution of (unobserved) private values without computing the Bayesian Nash equilibrium strategy or its inverse explicitly.

The main advantage of this method is therefore that it does not require to solve the differential equation (or system of equations) that characterize the Bayesian Nash equilibrium, and thus it is not necessary to have an analytical solution for the underlying theoretical model. Moreover, given the nonparametric nature of this method, the entire procedure is not subject to parametric assumptions and does not restrict a priori the function $F(.)$ to belong to some class of specific distributions. Another advantage of the indirect procedures is that they allow the derivation of important identification results and testable restrictions coming from game theory that can be used to validate the model.

Based on the equation that defines the inverse of the equilibrium strategy, the authors show that the model is nonparametrically identified in an IPV framework. Other papers by the same authors and others consider other auction models in a similar fashion. Including the affiliated private value model, models with asymmetric bidders, dynamic auction models and model with risk averse bidders. Laffont and Vuong (1996) generalize the

[13] In the empirical application, Laffont, Ossard, and Vuong (1995) use a log-normal distribution.

identification result in Guerre, Perrigne, and Vuong (2000) to symmetric APV models.[14]

The method in Guerre, Perrigne, and Vuong (2000) calls for a two step procedure. In the first step, a sample of pseudo private values is obtained while using nonparametric estimators for the distribution and density of observed bids. With this pseudo sample, the second step consists of estimating the density of bidders' private values nonparametrically. This estimator is shown to have desirable properties such us uniform consistency and the achievement of the optimal convergence rate by appropriate choice of vanishing rates for bandwidths.

IV. Nonparametric Estimators

For the last sixty years the statistical literature on nonparametric methods has developed considerably. This methodology proves to be especially useful in cases, in which one has no precise information about the form and class of, e.g., the true density of a random variable. The histogram is one of the oldest nonparametric methods for density estimation; it has the disadvantage of being discontinuous and too “rough”. There are several other methods available in this literature, such as kernels, splines, nearest-neighbor and local polynomials (see Härdle (1991) and Pagan and Ullah (1999) for a comprehensive discussion). By far, the most widely used is the kernel method.

Every method has some cost associated with it. The major problem cited in the literature faced by nonparametric procedures is the “curse of dimensionality”. The precision of the estimator (exponentially) deteriorates when the number of variables increases. That is, large data sets are needed to get accurate estimators.

IV.1. Kernel Density Estimators

The idea behind the estimation of a density, $F(.)$, using kernel estimation is very similar to the histogram. However, in kernel estimation one averages over kernel functions instead of averaging over data points.

[14] Laffont and Vuong (1996) explicitly acknowledge that their identification result is a generalization of a result in a previous version of Guerre, Perrigne, and Vuong (2000).

More formally, let X_1, X_2 ... be a sequence of random vectors in $\mathbb{R}^p$, where each X_i is distributed as $F(.)$ with density $f(.)$. A *kernel density estimator* of the density $f(.)$ at $x \in \mathbb{R}^p$ is defined as

$$\hat{f}(x) = \frac{1}{nh^p} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$$

where $h > 0$ is a bandwidth and $K(.)$ is a kernel (see Parzen (1962) and Rosenblatt (1956)). The bandwidth h is a smoothing parameter which regulates the degree of smoothness of the estimator.

The following assumptions are sufficient to obtain a pointwise consistent estimator,

1. The kernel $K(.)$ is a bounded, even and integrable function from $\mathbb{R}^p$ to $\mathbb{R}$ with $\int K(x)\, dx = 1$. In particular, because $K(.)$is bounded, $\hat{f}(x)$ has finite moments of any order.
2. The random vectors X_1, X_2 ... are independent and identically distributed.
3. $f(.)$ is continuous at x.
4. $K(.)$ is a Parzen-Rosenblatt kernel. More precisely, $K(.)$ satisfies

$$\lim_{\|x\| \to \infty} \|x\|^p K(x) = 0,$$

where $||.||$ is the Euclidean norm on $\mathbb{R}^p$. For instance, a function $K(.)$ with bounded support satisfies this condition.[15]

5. $h = h_n$, where $\{h_n\} \subset \mathbb{R}_+$ is a nonstochastic sequence satisfying

a) $h_n \to 0$ and

b) $nh^p \to \infty$, as $n \to \infty$

The kernel estimator $\hat{f}(.)$ has (further) desirable statistical properties. The asymptotic distribution of the estimator has been derived under additional regularity conditions. Also, uniform consistency has been established by strengthening some of the underlying assumptions. See e.g., Silverman (1986) or Bierens (1983) for a rigorous treatment and proofs of all these results mentioned above and the assumptions needed in each case.

[15] This assumption can be weakened if one is willing to make global assumptions on $f(.)$ such as it is bounded over $\mathbb{R}^p$.

As is well-known in the statistical literature, nonparametric estimators attain a lower convergence rate than parametric ones. For the kernel density estimator the best (uniform) rate of convergence has been established by Stone (1982) and it is given by $r^{*} = (n/\log n)^{\{R/(2R+p)\}}$, where R is the "degree" of smoothness of $f(.)$.

Next, I provide expressions for the bias and variance of this estimator.

Bias of $\hat{f}(x)$:

$$\mathrm{E}[\hat{f}(x)] - f(x) = \frac{h^R}{R!}\left[\int M_x^{(R)}(0,u)K(u)du + o(1)\right]$$

where

$$\int M_x^{(R)}(0,u)K(u)du = (-1)^R \sum_{i_1=r}^{p} \cdots \sum_{i_r=1}^{p} \frac{\partial^r f(x)}{\partial x_{i_1} \ldots \partial x_{i_r}}.$$

Variance of $\hat{f}(x)$:

$$\mathrm{Var}[\hat{f}(x)] = \frac{1}{nh^p}\left[f(x)\int K^2(u)du + o(1)\right].$$

IV.2. Local Polynomial Fitting

In this section I describe the main characteristics of local polynomial estimators.[16] I present the estimator for the case of one-dimensional explanatory variables $X_1, \ldots, X_n$.

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$, be an independent and identically distributed sample from a population (X, Y). The objective is to estimate the regression function $m(x_0) = E(Y|X = x_0)$ and its derivatives $m'(x_0), m''(x_0), \ldots, m^p(x_0)$. Assume that the $(\rho + 1)$st derivative of $m(.)$ exists at the point x_0. Using a Taylor expansion for x in a neighborhood of x_0, the regression function $m(.)$ can be locally approximated by a polynomial of order ρ, that is

$$m(x) \approx m(x_0) + m'(x_0)(x - x_0) + \frac{m''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{m^{(\rho)}(x_0)}{\rho!}(x - x_0)^{\rho} \quad (9)$$

[16] The discussion in this section follows closely Fan and Gijbels (1996). The reader is referred to this text for further details.

In terms of a weighted least squares (LS) regression problem one can write

$$\min_{\beta} \sum_{i=0}^{n} \{Y_i - \sum_{j=0}^{\rho} \beta_j (X_i - x_0)^j\} \frac{1}{h} K_h(X_i - x_0) \tag{10}$$

where h is a bandwidth and $K_h(.) = K(./h)/h$ with K a kernel function assigning weights to each observation.

Let $\hat{\beta}_j$, $j = 1, \ldots, \rho$ be the solution to the LS problem (10). By comparing the Taylor expansion in (9) to the LS problem (10) it is clear that $\widehat{m}_\nu\ (x_0) = \nu!\, \hat{\beta}_\nu$ is an estimator for $m^{(\nu)}(.)$, $\nu = 0,1, \ldots, \rho$.

To derive an expression for $\hat{\beta}$ it is more convenient to work with matrix notation. Let X denote the matrix of regressors of problem (10) and organize the variables Ys and the estimators $\hat{\beta}_j$ in column vectors, i.e.

$$X = \begin{pmatrix} 1 & (X_1 - x_0) & \ldots & (X_1 - x_0)^{\rho} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (X_n - x_0) & \ldots & (X_n - x_0)^{\rho} \end{pmatrix}, \ y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \text{ and } \hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \vdots \\ \hat{\beta}_{\rho} \end{pmatrix}.$$

Further, let W be the $n \times n$ diagonal matrix of weights: $W = diag\{K_h(X_i - x_0)\}$. Then the weighted LS problem above can be written as

$$\min_{\beta} (y - X\beta)^T W (y - X\beta),$$

with $\beta = \left(\beta_0, \ldots, \beta_\rho\right)^T$. From LS theory the solution to this problem is given by

$$\hat{\beta} = (X^T W X)^{-1} X^T W y$$

Like in the case of kernel estimators there are expressions for the (conditional) bias and variance of the LPE. The following theorem by Ruppert and Wand (1994) establishes the asymptotic expansions for the bias and variance of the estimator $\widehat{m}_\nu(x_0) = \nu!\,\hat{\beta}_\nu$. First I need to introduce the following notation. Let $\mu_j = \int u^j K(u)\,du$ and \$$\nu_j = \int u^j\ K^2(u)\,du$. Also let $S = \left(\mu_{\{j+l\}}\right)_{\{0\le j,l\le\rho\}}$, $\tilde{S} = \left(\mu_{\{j+l+1\}}\right)_{\{0\le j,l\le\rho\}}$, $S^* = \left(\nu_{j+l}\right)_{\{0\le j,l\le\rho\}}$, $c_p = \left(\mu_{\rho+1},\dots,\mu_{2\rho+1}\right)^T$, $\tilde{c}_\rho = \left(\mu_{\rho+2},\dots,\mu_{2\rho+2}\right)^T$. Further, consider the unit vector \$$e_{\nu+1} = (0,\dots,0,1,0,\dots,0)^T$, with 1 on the $(\nu+1)$st position.

Theorem [Ruppert and Wand (1994)]: *Assume that $f(x_0) > 0$ and that $f(.)$, $m^{\rho+1}(.)$ and $\sigma(.)$ are continuous in a neighborhood of x_0. Further, assume that $h \to 0$ and $nh \to \infty$.*

Then the asymptotic conditional variance of $\widehat{m}_\nu(x_0)$ is given by

$$\mathrm{Var}(\hat{m}_\nu(x_0)|X) = e_{\nu+1}^T S^{-1} S^* S^{-1} e_{\nu+1} \frac{\nu!^2\sigma^2(x_0)}{f(x_0)nh^{1+2\nu}} + op\left(\frac{1}{nh^{1+2\nu}}\right)$$

The asymptotic conditional bias for $\rho - \nu$ odd is given by

$$\mathrm{Bias}(\hat{m}_\nu(x_0)|X) = e_{\nu+1}^T S^{-1} c_\rho \frac{\nu!}{(\rho+1)!} m^{(\rho+1)}(x_0) h^{\rho+1-\nu} + op(h^{\rho+1-\nu})$$

Further, for $\rho - \nu$ even the asymptotic conditional bias is

$$\begin{aligned}\mathrm{Bias}(\hat{m}_\nu(x_0)|X) = {} & e_{\nu+1}^T S^{-1} \tilde{c}_\rho \frac{\nu!}{(\rho+2)!}\{m^{(\rho+2)}(x_0) + (\rho+2)m^{(\rho+1)}(x_0)\frac{f'(x_0)}{f(x_0)}\}h^{\rho+2-\nu} \\ & + op(h^{\rho+2-\nu})\end{aligned}$$

provided that $f'(.)$ and \$$m^{(p+2)}(.)$\$ are continuous in a neighborhood of x_0 and $nh^3 \to \infty$.

From the above theorem it is clear that there is a theoretical distinction between the cases $p-\nu$ odd and $p-\nu$ even. Moreover, polynomial fits with $p-\nu$ odd outperform those with $p-\nu$ even. For an exhaustive and detailed discussion see Fan and Gijbels (1996).

As with kernel estimators the crucial choice for LPEs is the bandwidth parameter. Of less importance (in practice) is the choice of the order of the polynomial, ρ and the kernel used for weighting. Optimal choices for bandwidths and kernels involved in LP estimation have been studied in the statistical literature. With respect to the choice of ρ, Fan and Gijbels (1996) emphasize that for many applications the "rule" $\rho = \nu + 1$ suffices.

LP fitting has a number of attractive features both from theoretical and practical point of view. The most relevant is the behavior of these estimators at the boundaries. Put in other words, the bias at the boundary stays automatically of the same order as in the interior, without the use of specific boundary kernels. This is remarkably different from kernel estimation (and other nonparametric methods as well).[17] On top of the advantages at the boundaries, LPEs have nice minimax efficiency properties; the asymptotic minimax efficiency for commonly used orders is 100% among all linear estimators and only a small loss has to be tolerated beyond this class.[18]

V. Collusion

In this section I present the relevant theoretical background to analyze collusive models and how the literature has managed the econometric problem derived from this theoretical framework. To make the presentation of the model more transparent I consider the two-bidder case here. Although this section explains the theoretical framework for an auction model, it could be that bidders compete for the right to sell their services and thus the winner is the bidder with the lowest bid. The main features of the auction model outlined here are the same for the procurement setting with the appropriate sign changes.

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[17] See Silverman (1986), Fan and Marron (1993)

[18] LPEs belong to the class of linear smoothers. This class includes the Nardaraya-Watson estimator, the Gasser-Müller estimator, orthogonal series estimators and spline smoothers.

Although collusion is an illegal activity, it is a pervasive problem in auction markets such as public construction, school milk supply, stamps; see (Comanor and Schankerman (1976), Feinstein, Block, and Nold (1985), Lang and Rosenthal (1991), Porter and Zona (1993), Bajari (2001), Porter and Zona (1999), Pesendorfer (2000), Asker (2010) and Harrington (2008)) and municipal bonds among others. In this section I briefly discuss the relevant theoretical and empirical work that has been done in an attempt to better understand and detect this practice. As emphasized by Baldwin, Marshall, and Richard (1997) the profitability and prevalence of bid rigging call for the incorporation of the possibility of collusive behavior into empirical models.

Theoretical work on bidder collusion at auctions is extensive. The following description is by no means exhaustive. Robinson (1985) analyzes the relative propensity of the different auction formats, second-price, first-price, and English auctions, to collusion. In particular, the author shows the relative no susceptibility of first-price auctions to bidder collusion. The analysis of collusion in second-price auctions was initiated by Graham and Marshall (1987) in a symmetric IPV model. Graham, Marshall, and Richard (1990) extend previous results to the case of distributionally heterogeneous bidders.[19] McAfee and McMillan (1992) analyze collusion in first price auctions by an all-inclusive coalition. They also study a pre-auction knockout mechanism used by the cartel. As is well-known, asymmetric first--price auction models yield an equilibrium with no closed form.[20] Marshall, Meurer, Richard, and Stromquist (1994) look at less than all--inclusive cartels at first--price auctions and propose numerical techniques to solve for the equilibrium of the underlying asymmetric game. From a more policy-oriented view, Marshall and Meurer (2004) argue that the relative lack of attention given to bidder collusion is based on the mistaken belief that the economics of bidder collusion and that of price fixing are essentially equivalent. The authors illustrate the differences between standard industry posted-price cartels and collusion by bidders at auctions or procurements by means of several models and examples. Moreover, they propose policy recommendations that apply specifically to bidder collusion. The article by Hendricks, Porter, and Tan

[19] Also, the work by Mailath and Zemsky (1991) is also another excellent reference for collusion in second– price auctions. I would like to thank a referee for pointing this out.
[20] Except for some special cases (see footnote 8).

(2008) extends the theory of legal cartels to affiliated private value and common value environments.

The empirical literature on bidder collusion is more limited. Hendricks and Porter's (1989) survey paper discusses mechanisms that are likely to facilitate collusion in auctions and propose some tests in order to detect bid rigging by analyzing two commonly used data sets within both the IPV framework and the Common Value (CV) framework. Porter and Zona (1993, 1999) and Pesendorfer (2000) concentrate on collusion in auction markets given that it is known that bid--rigging has taken place. The objective of these papers is basically to find empirical facts in collusive markets.

Another set of empirical papers proposes methods to detect collusion. Porter and Zona (1993), Baldwin, Marshall, and Richard (1997) and Bajari and Ye (2003) study collusion in IPV settings. The paper by Asker (2010) seeks to better understand the functioning of an operating cartel. Within the structural approach, the author examines a first--price knockout auction mechanism used by a cartel of stamp dealers in the 1990s.

Porter and Zona (1993) argue that detection of collusion is possible because of limited participation in the collusive setup. Accordingly, they attempt to detect differences in behavior between ring members and non-members. The authors have detailed information of the operation of a cartel and its bidding practices. In particular they study the bidding behavior of firms competing for highway construction projects on Long Island in the early 1980s. They propose two types of analysis. The first one is based on the level of bids and the second one on the ranking of bids. Accordingly, Porter and Zona (1993) argue that the evidence of collusive behavior relies on the fact that the lowest non-cartel bidder's behavior is not statistically different from that of other non-cartel firms, while the determinants of the low cartel bid differ from those of higher cartel bids. By knowing the identities of cartel members Porter and Zona (1993) estimate two models for each subgroup of bidders and test the null hypothesis of absence of collusion by testing the equality of parameter values in the models. The starting point for each (econometric) model is given by the first order conditions for an equilibrium strategy, namely

$$\varphi_{i\ell} + (b_{i\ell} - c_{i\ell})\varphi'_{i\ell}(b_{i\ell}) = 0 \tag{11}$$

where b_{il} is the submitted bid for firm i in project l, c_{il} is the corresponding cost, and φ_{il} is the probability of winning.

Regarding the analysis based on the level of bids, Porter and Zona (1993) exploit the characterization of the equilibrium bid given by (11) and assume that equilibrium behavior satisfies the log-linear bidding rule

$$\log(b_{i\ell}) = \alpha_\ell + \beta' X_{i\ell} + \epsilon_{i\ell} \tag{12}$$

where α_l is an auction-specific effect, X_{il} is a vector of observable variables affecting firm i's probability of winning object l. In the empirical application the authors include the utilization rate, the firm's backlog and capacity and a dummy variable regarding the location of the firm. The error term, ϵ_{il}, represents private information for firm i on project l. It is assumed to have zero expectation and an auction-specific variance, σ_l^2.

By estimating the auction-specific variance using the auction mean-squared residual, the authors implement a feasible generalized least squares (GLS) estimator to obtain estimates for the parameters in (12). The reported results are given for three subsets of data: bids from all firms, bids from competitive firms and bids from cartel firms only. The authors conjecture that if all bids were competitive, the three subsets of data should give the same underlying parameters. On the other hand, if cartel bids were not competitive, then the model would be misspecified, and only the estimators based on competitive data would be consistent.

The two main conclusions from this analysis indicate that the model fits the competitive data reasonably well according to a Wald test and that bids from cartel firms statistically differ from those of competitive firms. The authors claim that the analysis based on the ranking of bids (i.e. the second kind of test proposed by the authors) sheds light on the reasons for this discrepancy.

To perform an analysis based on the ranking of bids, the main argument used by Porter and Zona (1993) states that fundamental differences may exist between the ordering of competitive and cartel bids conditional on observed data. This observation relies on the fact that firms submitting phantom bids know that a designated firm will submit a lower bid (recall that Porter and Zona (1993) study a procurement-auction). Thus, complementary bids have no probability of winning by design. The rationale for phantom bidding is just to create the appearance of competition. However, the designated cartel bid must bid competitively like the remaining non-cartel firms. The authors do not explain how the designated bidder is selected.

From this observation, Porter and Zona (1993) propose a rank-based test intended to detect differences in the ordering of higher bids, as opposed to the determinants of the probability of being the lowest bid, for each set of firms. The conjecture in this case is that if cartel bids were indeed competitive, their ordering should reflect observable cost differences.

To implement the rank test the authors use equation (12) to characterize the probability of winning by approximating it with a multinomial logit (MNL) model as follows

$$\ln \mathrm{P}[b_{i\ell} < \min_{j \neq i} b_{ij}] = \theta_\ell + \beta' \frac{X_{i\ell}}{\sigma_t} \sqrt{\frac{\pi}{6}}$$

Let $-X_{il}/\sigma_t = Z_{il}$. The MNL model giving the log probability that firm i will win auction l is,

$$\ln \mathrm{P}[b_{i\ell} < \min_{j \neq i} b_{ij}] = \alpha_\ell + \beta' Z_{i\ell} \quad (13)$$

By exponentiation of the log probabilities, equation (13) can be expressed as

$$\mathrm{P}[b_{i\ell} < \min_{j \neq i} b_{ij}] = \frac{\exp(\beta' Z_{i\ell})}{\sum_j \exp(\beta' Z_{j\ell})}$$

Given the MNL specification chosen by the authors, the probability of observing any particular ranking of bids on a project can be expressed as the product of individual choice probabilities. If n_l bids are submitted on job $\el, $l = 1, \ldots L$, the likelihood of observing the rankings of the data from all auctions in the sample is

$$L(\beta) = \prod_{t=1}^{L} \prod_{i=1}^{n_\ell} \frac{\exp(\beta' Z_{r_i \ell})}{\sum_j \exp(\beta' Z_{r_j \ell})}$$

where r_m denotes the index of the firm with bid ranked m (see Porter and Zona (1993) for further details).

The model is estimated using standard maximum likelihood (ML) estimation for MNL. If the model is correctly specified, the parameters can be estimated from any subset of the data. To test the hypothesis of no phantom bidding the authors use a Likelihood Ratio (LR) test. In other words, the null hypothesis states that the parameters estimated using only the lowest cartel ranks and those estimated from higher cartel ranks should be the same. As pointed out in the paper, rejection of the null hypothesis could be because of two reasons. First, the model may be misspecified for some reason other than phantom bidding. However, the authors argue that if the test did not reject the null when applied to competitive data, then it is less likely to have a specification problem. The second reason leading to rejection could be due to an effect that is common to non-winning cartel bids but not non-winning competitive bids. Porter and Zona (1993) conclude that under the assumptions of the model, the rejection is likely to be the result of phantom bidding.

The main conclusion drawn from the analysis of competitive bid rank data states that the same process generates these bids whether or not they are low. In other words, the estimates are stable over ranks and the LR test does not lead to rejection of the null hypothesis of no model misspecification. The second analysis based on cartel bid rank data yields the opposite conclusion, namely that cartel bids are generated by a different process depending on whether or not they are low.

Finally, the authors conclude that they have found evidence supporting cartel activity in the sample since they do not have reason to believe that the difference between cartel and competitive bidding is structural.

Bajari and Ye (2003) propose a model in which bidders are asymmetric in a procurement first-price-auction setup. The authors derive two conditions that must hold under competitive bidding, namely, conditional independence of bids and exchangeability of bid distributions. They also propose a third test based on Bayesian techniques which requires inside information from the industry. Bajari and Ye (2003) apply their tests to a data set on seal coat contracts in the Midwest.

I now explain this set of conditions in more detail.

Let $Z = (Z_1, Z_2, , \dots, Z_N)$ denote a set of covariates that is observable to all firms. Let $G_i(.; Z)$ be the cumulative distribution function of firm i's bid given covariates. Observe that the distribution of bids depends on the entire vector Z.

Conditional on $Z = z$, firm i's bid and firm j's bid are independently distributed. As a result

$$G(b_1, \dots, b_N; z) = \prod_{i=1}^{N} G_i(b_i; z), \tag{14}$$

where $G(b_1, \dots, b_N; z)$ is the joint distribution of bids. As mentioned in Bajari and Ye (2003) there is more than one way of testing this condition. Ideally each side of (14) can be estimated nonparametrically and compared. However with limited data this becomes less attractive. Alternatively, regression-based methods can be used. That is the marginal distribution, $G_i(b_i; z)$, can be estimated using a regression (see also Porter and Zona (1993, 1999)), and then the residuals are tested to assess if they are independent.

The second condition that must hold in equilibrium when bidding is competitive is referred to as exchangeability of the distribution of bids. More formally, let π be a permutation, that is, a one-to-one mapping from the set $\{1, \dots, N\}$ onto itself. Then exchangeability is defined as follows: for any permutation π and any index i the following equality must hold

$$G_i(b; z_1, z_2, z_3, \dots, z_N) = G_{\pi(i)}(b; z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)}, \dots, z_{\pi(N)})$$

Like for conditional independence, regression-based methods can be used to test this condition. This is the approach taken in Bajari and Ye (2003).

The papers by Baldwin, Marshall, and Richard (1997) and Asker (2010) use a structural approach to analyze auction data.[21] Baldwin, Marshall, and

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[21] In the following section I explain in more details the structural approach used in the literature as opposed to reduced form approach.

Richard's (1997) data set comes from oral ascending auctions. Therefore, the authors concentrate on this auction format to derive the econometric models used in the application. Moreover, one of the maintained assumptions in that paper is distributional homogeneity across bidders' valuations (i.e. bidders are symmetric). At a more technical level, the empirical model is fully parametric. On the other hand, the work by Asker (2010) considers asymmetries across bidders. However, its main objective is to analyze the functioning of a cartel as opposed to study the main auction. The knockout auction is conducted using a sealed-bid format. The author focuses in modeling the pre-auction knockout mechanism used by the ring to designate the serious bidder at the main auction. The econometric procedure used is fully nonparametric.

Baldwin, Marshall, and Richard (1997) formulate various empirical models using the structural approach allowing for both bidder collusion and supply effects in order to analyze auctions for timber in the Pacific Northwest. The main objective is to determine whether price variations, conditional on demand characteristics, are better explained by collusion or, alternative, variations in timber supply conditions. The authors provide some evidence revealing the similarity between the winning bid and the reserve price in timber auctions during the late 1970s and early 1980s. This information motivates the following observation stated in the paper. *"Although effective bidder collusion will produce winning bids that are low relative to the predictions of a suitable model of non-cooperative behavior, it clearly would not be reasonable to conclude that bidders are colluding solely on the basis of the observation of relatively low winning bids"*. Accordingly, five models are estimated: the non-cooperative model with no supply effects, the collusive model with no supply effects, the non-cooperative model with supply effects and two nested models that contain both collusion and supply effects.

I concentrate here on the collusive model with unit supply. The underlying theoretical model that leads to the empirical model is based on Graham and Marshall's (1987) collusive mechanism. In this pre-auction knockout, colluding bidders find participation individually rational. The effective coalition size is denoted by K_i. Conditional on K_i, the price of the object is given by the $K_i +$ 1st order statistic of the private values. Two important assumptions behind the model are that bidder collusion is a (symmetric) bidder-specific decision and that there is only one coalition. Thus all non-ring bidders act non-cooperatively. Another important element of the model is the probability of joining the coalition, p_i. Conditional on $Z_i = z_i$ the natural logarithm of private values is assumed to be normally distributed with mean $\beta' z_i$ and variance σ^2, where z_i is a vector of covariates. The standardized price, $U_i = (ln(B_i/v_i) - \beta' z_i)/\sigma$, is a mixed random variable with $Pr(U_i = t_i) = p_i^{n_i}$ and density $h_c(.|z_i)$, where B_i is the winning bid at auction i, v_i is the volume in mbf and $t_i = (ln(r_i/v_i) - \beta' z_i)/\sigma$, with r_i denoting the reserve price.[22] The authors specify the likelihood of the collusive model as well as a parametric expression for p_i. Let C_i be the coalition participating at auction i and $y_i = (ln[1.055\ (r_i/v_i)] - \beta' z_i)/\sigma$. The likelihood function is

$$L(\beta, \gamma, \sigma; d) = \left[\prod_{i \in I_a} \frac{v_i}{\sigma b_i} h_c(u_i|z_i)\right] \left[\prod_{i \in I_b} H_c(y_i|z_i)\right]$$

where $I = I_a \cup I_b$ denotes the total number of observations. I_b is the set containing 13 observations in which the winning bid is within 5.5% of the reserve price, the remaining observations belong to I_a.[23] γ is a parameter from the expression for p_i and d denotes the data set.

The model is estimated using standard ML techniques. The main conclusion from this analysis is that the collusive model outperforms the non-cooperative model. Moreover, the authors highlight that both models pass the Kolmogorov-Smirnov test. Thus, log-normality is not rejected. This leads the authors to the further conclusion that the increase in the log likelihood function observed in the collusive case is not due to misspecification of the private value distribution. With respect to the model containing supply effects and the nesting models, the authors conclude that as soon as collusion is taken into account, supply does not add explanatory power. Overall, the collusive model emerges as the preferred model.

More recently Aryal and Gabrielli (2013) propose a two-step procedure to detect collusion in asymmetric first-price procurement (auctions). First, they use a reduced form test to short-list bidders whose bidding behavior is at-odds with competitive bidding; and Second, they estimate the (latent) cost for these

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[22] In their paper Baldwin, Marshall, and Richard (1997) provide the explicit forms for the density and cumulative distribution functions of U_i.

[23] The13 observations in the set I_b are considered as outliers. However, Baldwin, Marshall, and Richard (1997) argue that it would be inappropriate to discard them from the estimation. See section VI.C and Appendix D of that paper for a detailed discussion.

bidders under both competition and collusion setups. Since, for the same bid the recovered cost must be smaller under collusion- as collusion increases the mark-up- than under competition, detecting collusion boils down to testing for first-order stochastic dominance, for which we the authors use the classic Kolmogorov-Smirnov and Wilcoxon-Mann-Whitney tests. The paper also presents Bootstrap based Monte Carlo experiments for asymmetric bidders that confirm that the procedure has good power to detect collusion when there is collusion. The authors illustrate the procedure by implementing the tests for Highway Procurement data in California and conclude that there is no evidence of collusion even though the reduced form test supports collusion. This highlights potential pitfalls of inferring collusion based only on reduced form analysis.

About how collusion is sustained, Aryal and Gabrielli (2013) assume that the bidding ring can control the bids of the members and can eliminate all ring competition and hence there is only one serious bidder, the most efficient bidder, i.e. there is efficient collusion. This is the most favorable environment for collusion and for the purpose of the paper it is not necessary to spell out the exact rules of sharing the surplus.[24]

VI. Conclusions

This paper is an attempt to survey the state of the art regarding First-Price Auction Theory as well as the Econometric Methods mainly used to estimate the theoretical models. The reason for focusing on sealed bid first price auction mechanisms is that most of the data sets available for empirical research come from this auction type. The central aspect of the paper is the so-called Structural Approach to analyze auction data. This approach is closely related to the underlying game theoretic model. Given the tight relation between the theoretic model and the corresponding econometric model I start by reviewing the different paradigms developed in the literature to model auctions. In this respect section 2 contains a description of the Private Value Paradigm and the Common Value Paradigm to model auctions, which can be further classified into symmetric and asymmetric models, and independent and affiliated models. All of these lead to different auction models. After characterizing each

[24] Marshall and Marx (2007) show that only in the first-price auction, if the ring cannot control the bids then the equilibrium entails multiple bids and the model need not be identified.

model and in particular how the Bayesian Nash equilibrium looks like in each case I have described the most important results concerning the identification issue. This key aspect is what then allows the researcher to propose a Structural Econometric Model. Section 3 then describes extensively the structural approach, which distinguishes two kinds of methods: direct methods and indirect procedures. Among direct methods, which were first developed in the literature, there are maximum-likelihood based procedures and simulation-based procedures. The last ones are more attractive from a computational point of view because they do not need to solve explicitly for the equilibrium of of the model, something that could be highly computationally demanding or even impossible in some cases. Later, Guerre, Perrigne, and Vuong (2000) introduced the use of indirect methods which rely on a very simple buy key observation, namely the relation between the bid distribution $G(.)$ and the private value distribution $F(.)$ through the strictly increasing equilibrium strategy. This observation allows identification and leads to a natural two-step procedure.

Given the nonparametric nature of indirect procedures, I have devoted a section to explain the use of kernel methods and local polynomials for estimating auction models.

Finally, given that one source of asymmetries that constitute a pervasive problem within auction markets is collusion this survey has also a section that presents the most relevant literature (theoretical and applied) on collusion.

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